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Boundary value problems with initial jumps for singularly perturbed integro-differential equations

ABSTRACT

of the PhD thesis for the degree of Doctor of Philosophy (PhD)
in the specialty «8D05401-Mathematics»

Relevance of research. Differential and integro-differential equations are the main tools for describing a wide range of processes in science and technology. A significant class of such equations are equations with small parameters. A small parameter is not just a small number, this is a key factor that changes the quality characteristics of the system. The presence of a small parameter causes deviations in the resulting equation. That is, the inclusion of a small parameter in the equation leads to significant changes in the properties of the solution and requires a special analysis when this parameter approaches zero.

Equations with a small parameter are divided into *regular* and *singular.*

In contrast to regularly perturbed problems, in singularly perturbed problems, when a small parameter tends to zero, an equation different from the original one appears, which is characterized by a change in order, violation of boundary conditions and the appearance of such features as boundary layers and jumps in the structure of the solution. Thus, the main difference between regularly and singularly perturbed problems is that at $\varepsilon \to 0$, the solution of a regularly perturbed problem converges to the solution of the unperturbed task in the entire given area. This allows us to apply the method of direct perturbation theory.

In general, the name of the perturbation theory indicates that even a small perturbation introduced into the equation can significantly affect the nature of the solution. Such equations are an important tool of mathematical modeling.

The dissertation is dedicated to the study of boundary value problems for singularly perturbed integro-differential equations of any order, in which a small parameter is not in front of one derivative, but in front of several derivatives.

Aim of the study. Study of the boundary value problem for a singularly perturbed linear integro-differential equation of higher order.

Object of study. Boundary value problems with initial jumps for singularly perturbed integro-differential equations.

Research methods. The method of boundary functions is used in the dissertation to find the solution of the boundary value problem for the singularly perturbed linear integro-differential equation.

Theoretical and practical significance. Dissertation is theoretical in nature. The obtained results make a significant contribution to the development of the theory of singularly perturbed integro-differential equations. This equation belongs to one of the most important classes of singularly perturbed equations. It is a mathematical model for studying various processes in physics, chemistry, biology, engineering and other fields.

This research complements and develops the previously existing theory of singular perturbation. Thus, this work is of great importance for the further development of asymptotic methods for the analysis of complex operator equations.

In addition, the obtained results can be used in educational programs for the training of specialists in the field of differential equations, applied mathematics, mechanics and mathematical modeling.

Scientific novelty. The main results presented to the defense. In the dissertation, previously studied problems were investigated and the following new results were obtained:

- obtained analytical formula for solving the boundary value problem for singularly perturbed linear integro-differential equations of the $m+n$-th order with small parameters before m higher derivatives;
- obtained asymptotic estimates for solving a singularly perturbed problem;
- a certain order of growth of the solution of the singularly perturbed problem and its derivatives;
- a modified undisturbed task is constructed;
- the obtained estimate of the difference between the solutions of the singular and modified undisturbed problem;
- the initial jump solutions and the integral term are found;
- the asymptotic expansion of the solution of the boundary value problem is constructed with any degree of accuracy under small parameter.

Approbation. Dissertation results were presented at the International Scientific Conference " Farabi World" (Almaty, Kazakhstan, April 6-8, 2021); International Scientific Conference "Farabi World" (Almaty, Kazakhstan, April 6-7, 2023); at the international scientific conference " V Borubaev Readings", devoted to the 70th anniversary of the National Academy of Sciences of the Kyrgyz Republic and the 40th anniversary of the Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic (June 20-21, 2024); "Computing and information technologies in science, engineering and education", devoted to the 90th anniversary of the Kazakh National University named after Al-Farabi and the 75th anniversary of Academician Sh. S. Smagulov (CITech-2024) (October 23-24, 2024); Mathematical Congress, Traditional International Mathematical April Conference dedicated to the Day of Scientists (April 3-7, 2026); a report was presented and discussed at the scientific seminar of the Department of Mathematics, Faculty of Mechanics and Mathematics of the Kazakh National University named after al-Farabi.

Reliability and validity of scientific statements, conclusions and results. The results obtained during the research work are confirmed by the data of previously recognized scientists, publications in indexed international journals, as well as conference materials.

Publication. Seven articles were published based on the results of the dissertation:

– Publications in journals included in the Web of Science and Scopus databases:

1. Dauylbayev M.K., Konisbayeva K.T. Asymptotical Convergence of Solutions of Boundary Value Problems for Singularly Perturbed Higher-Order Integro-Differential Equations / Quaestiones Mathematicae Published online: 5 Jul V47, 2024 https://doi.org/10.2989/16073606.2024.2364297 Квартиль:Q2 Процентиль: 54, Индекс цит.: 1.4, SJR: 0.44.

2. Dauylbayev M.K., Konisbayeva K.T. Asymptotic expansion of the solution of a boundary value problem for singularly perturbed integro-differential equations of higher orders\ Boundary Value Problem Journal \ 2025 (WoS (SCIE) Q2) https://doi.org/10.1186/s13661-025-02140-5 Квартиль: Q1, Процентиль: 75, Индекс цит.: 2, SJR: 0.479.

3. Mirzakulova A.E., Dauylbayev M.K., Konisbayeva K.T. Asymptotic estimations of the solution of a singularly perturbed equation with piecewise constant argument Lobachevskii Journal of Mathematics/45(4), 2024г. DOI:10.1134/S1995080224601450 Квартиль: Q2, Процентиль: 57, Индекс цит.: 1.5, SJR: 0.435 [72]

4. Mirzakulova A.E., Konisbayeva K.T. Uniform asymptotic expansion of the solution for the initial value problem with a piecewise constant argument / Bulletin of the Karaganda, «Mathematics Series».- 116(4), 2024 г. https://doi.org/10.31489/2024m4/138-148 Квартиль: Q2, Процентиль: 53, Индекс цит.: 1.4, SJR: 0.392.

5. Atakhan N., Nurpeissov K.S., Konisbayeva K.T. Asymptotic estimations of the solution for a singularly perturbed equation with unbounded boundary conditions \ Bulletin of the Karaganda, «Mathematics Series».- №4(108)/2022 https://doi.org/10.31489/2022m4/20-33 Квартиль: Q2, Процентиль: 53, Индекс цит.: 1.4, SJR: 0.392.

6. M.K. Dauylbayev, K.T. Konisbayeva, N.R. Tortbay Integral BVP for singularly perturbed system of differential equations // International Journal of Mathematics and Physics 12, No1, 25 (2021). С. 25-33. DOI:https://doi.org/10.26577/ijmph.2021. v12.i1.04 Квартиль:Q4, Процентиль: 7, Индекс цит.: 0.157, SJR: 0.4.

7. K.T. Konisbayeva, M.K. Dauylbayev, Mirzakulova A.E., Asymptotic estimates of the solution of the BVP for singularly perturbed integro-differential equations// Journal of Mathematics, Mechanics and Computer Science. №1(129)-2026. – P. 39-51. https://doi.org/10.26577/JMMCS129120264 Квартиль:Q4, Процентиль: 18, Индекс цит.: 0.157, SJR: 0.4.

– Publications in collections of materials of international scientific conferences:

1. Integral boundary value problem for a system of differential equations with singular perturbations // International scientific conference "Farabi world", Almaty, April 7-8, 2021: collection of materials (Almaty - 2021. -19 p.).

2. Boundary value problem for a singularly perturbed integral-differential equation with a small parameter in higher derivatives // "Farabi World" International Scientific Conference, Almaty, April 4-6, 2024: Materials (Almaty - 2024. - 21 p.).

3. Asymptotic estimation of the solution of boundary value problems for singularly perturbed integral-differential equations // International scientific conference devoted to the 70th anniversary of the National Academy of Sciences of the Kyrgyz Republic and the 40th anniversary of the Mathematical Institute of the National Academy of Sciences of the Kyrgyz Republic, "V Borubaev Readings", Bishkek, Kyrgyzstan, June 20-21, 2024: Material. (Bishkek, 2024. 59 p.)

4. Asymptotic decomposition of singularly perturbed integro-differential equations // Dedicated to the 90th anniversary of KazNU. Al-Farabi and the 75th anniversary of Academician Sh. S. Smagulov "Computing and information technologies in science, engineering and education" (CITech-2024), Almaty, Kazakhstan, October 23-24, 2024: Materials of the conference. (Almaty-2024. 89 p.)

5. Asymptotic estimation of the solution of a boundary value problem for a singularly perturbed integro-differential equation // Mathematical Congress-2026. Traditional International April Mathematical Conference dedicated to the Day of Scientists, Shymkent, Kazakhstan, April 3-7, 2026: collection of theses reports.

As a result of the dissertation research, 5 papers were published, including one article in the first (Q1) quartile in the journal "Boundary Value Problem" (75th percentile in the Scopus database), 4 articles in journals in the second (Q2) quartile: "Quaestiones Mathematicae" (54th percentile in Scopus), "Lobachevskii Journal of Mathematics" (57th percentile in Scopus) and "Bulletin of the Karaganda", "Mathematics Series" (53rd percentile in the Scopus database) and 2 articles in journals included in the fourth (Q4) quartile: International Journal of Mathematics and Physics (7th percentile in the Scopus database) and Journal of Mathematics, Mechanics and Computer Science (18th percentile in the Scopus database); - 5 publications in materials of international conferences.

Dissertation of structure. Dissertation consists of normative references, introduction, two main sections (each of which includes subsections), conclusion and list of references. The volume of the dissertation is 85 pages.

Content dissertation. In the introduction of the dissertation, the relevance of the chosen topic is substantiated, its scientific novelty is indicated, the research methods used are described, the theoretical and practical significance of the obtained results is indicated, and a brief overview of the structure of the work is given.

In the first section, we consider a two-point boundary value problem of the following form for a singularly perturbed linear integro-differential equation of order n+m with small parameters before m senior derivatives, provided that the roots of the additional characteristic equation are negative:

$$L_\varepsilon y = \sum_{r=1}^{m} \varepsilon^r A_{n+r}(t) \frac{d^{r+n} y}{dt^{r+n}} + \sum_{k=0}^{n} A_k(t) \frac{d^k y}{dt^k} = F(t) + \int_0^1 \sum_{j=0}^{n_1+p} H_j(t,x) y^{(j)}(x,\varepsilon) dx \quad (1)$$

$$\begin{cases} \left.\dfrac{d^i y}{dt^i}\right|_{t=0} = \alpha_i, \quad i = \overline{0, l-1} \\ \left.\dfrac{d^i y}{dt^i}\right|_{t=1} = \beta_i, \quad i = \overline{0, q-1} \end{cases} \tag{2}$$

where $\varepsilon > 0$ – small parameter, α_i, β_i – khowns constants, $A_n(t) \neq 0$, $A_{n+m}(t) = 1$, $m + n = l + q$, $p = \overline{1, m}$.

According to equation (1), the fundamental system of solutions, the Cauchy function and the boundary functions of the homogeneous singular perturbed differential equation are constructed, and their asymptotic estimates are obtained. Using known Cauchy function and boundary functions, the theorem for the analytical formula for solving the boundary value problem (1), (2) is derived and proved. Using an analytical formula, the theorem on the asymptotic estimation of the solution is proved. A modified undisturbed boundary value problem is constructed, consisting of initial jump solutions and an integral term. As a result, the limit transition theorem is derived and proved. To confirm the theoretical result, a numerical example obtained by modeling is given.

In the second section, the algorithm of asymptotic expansions with the accuracy of a small parameter for solving the boundary value problem (1), (2) is presented, and the obtained result is formulated and proved in the form of a theorem.

In the final section, an analysis of the general results is carried out.